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LETTER TO THE EDITOR

Scaling behaviour in discrete traffic models

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Abstract. We investigate the noisy, discrete, single lane highway traffic model proposed by Nagel and Schreckenberg and demonstrate by computer simulations that, as a function of the car density, there is a *dynamical* transition from laminar flow to jammed traffic in the system related to the divergence of the relaxation time of average car velocity. The critical density is close to the position of the maximum in the fundamental diagram. We give estimates of the critical exponents related to this jamming transition. The critical density of the *geometrical* ((1 + 1)-dimensional percolation) transition of jammed regions can be rather far from the jamming point depending on the strength of the noise which indicates that cooperativity leading to the jamming transition can be long range in character.

Numerous articles have been published in the last couple of years investigating discrete models of highway traffic flow [1–4]. It has been suggested that very simple probabilistic models based on cellular automata can reproduce features of real traffic, including a supposed transition from low-density laminar flow to a high-density phase, where start–stop waves are dominant. The behaviour of these simple models is very complex near this transition and, up to now, is still not well understood. In this letter, we would like to shed some light on what we believe to be the dynamical jamming transition with a well-defined critical point, as distinguished from the percolation-type transition of jammed regions in the same model.

We consider the model proposed in [1], a cellular automaton consisting of a one-dimensional array of cells with periodic boundary conditions. Every cell has $(v_{\max} + 2)$ states: it can be empty or it can contain a car with velocity $v = 0, 1, \ldots, v_{\max}$. We perform the following steps in parallel for all cars:

Acceleration: increase v by 1 if possible. (1*a*)

Deceleration: decrease v to avoid crash with the car in front. (1*b*)

Randomization: decrease v by 1 with probability p if possible. (1*c*)

Movement: move forward v sites. (1*d*)

The choice of $v_{\max} = 5$ is traditional in the literature of this model, and it can be considered as a limit speed. We would like to stress the importance of the third step. The fact that the model uses *breaking noise* is crucial. One could equally introduce random accelerations, but it can be shown that this type of perturbation dies out very quickly, see [5]. Throughout this letter we use $p = 0.25$ unless stated otherwise.

A convenient way to investigate the model is to draw a diagram of flow against density, the so-called *fundamental diagram*. It is a curve with a well-defined maximum near a density of 0.11. The occurrence of density waves is related to the nonlinearity of the fundamental diagram and it is expected that the jamming transition will occur somewhere near the

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maximum. At low densities the flow is 'free' with very few waves due to fluctuations (1*c*), which die out quickly, at high densities above the maximum start–stop waves dominate the system; it is in the 'jammed' state.

What is the nature of this transition? It has been suggested [6] that, approaching from the low-density region, the jammed regions grow in space and time and at the jamming point they form a $(1+1)$-dimensional interconnected infinite network. This would point to a percolation-type picture for the process.

We would like to investigate the dynamics without explicitly referring to the density waves by searching for *critical slowing down*. In order to do so we measure the average velocity $\bar{v}$ of all cars as a function of time t starting from randomly positioned cars with zero initial velocity ($\bar{v}(t=0)=0$). The function $\bar{v}(t)$ is expected to be monotonically increasing and asymptotically approaches the steady-state value v_∞. The small deviations in the measured curves from this expectation are due to fluctuations in finite-size samples which can be suppressed by averaging over an ensemble.

At early times and low densities the function $\bar{v}(t)$ is independent of the interaction between the cars and it follows the curve $v^*(t)$ for the zero-density, non–interacting limit ((1*b*) is disregarded): $\lim_{t\to 0}\bar{v}(t)=(1-p)t=v^*(t\leqslant 4)$. Gradually the dynamics are influenced by the car–car interaction and this is manifested in the slower convergence and in the lower asymptotic steady-state value v_∞.

In order to characterize the dynamics we define the *relaxation time*

$$\tau(\rho)=\int_0^\infty (\min(v^*(t),\langle\bar{v}_\infty\rangle)-\langle\bar{v}(t)\rangle)\,\mathrm{d}t \tag{2}$$

where the $\langle\cdots\rangle$ denotes ensemble average. Due to the dimensionless units no normalization was introduced. We use definition (2) because it expresses the two mechanisms of relaxation: the fast, essentially non-interacting part and the much slower part where interaction and jamming becomes important. The time-scale of the second process can be magnitudes larger than the first one because the emerging jams have to disperse.

We expect that at the jamming transition the global character of the flow should change and the approach of the steady state takes an extremely long time. Much below the transition emerging jams are independent: they disappear rather quickly. As the density is increased the cars escaping from one jam are getting collected by another (not necessarily nearly positioned) wave thus the jams interact and a complicated web of waves characterizes the steady state. This picture suggests that, in the thermodynamic limit, the relaxation time should diverge at the transition.

For finite sizes the divergence of the relaxation time at the critical density ρ_c is reflected in a peak of the $\tau(\rho)$ curve. In figure 1 we show data for system sizes $L=1024$, 2048, 4096, 5824 and 8192. We used a site-oriented multispin coding technique [4]. A relaxation data point in figure 1 for system size $*$ number of runs $=8000$ took 20 minutes CPU time on the CM-5 with 64 nodes, and approximately 1 day on an HP workstation.

The results can be interpreted in terms of critical slowing down and a finite-size scaling assumption is compatible with our data. Denoting the maximal value of τ for a given size L by $\tau_m(L)$ and the width of the peak at $\tau_m(L)/2$ by $\sigma(L)$:

$$\tau_m(L)\propto L^z \tag{3a}$$

$$\sigma(L)\propto L^{-1/\nu} \tag{3b}$$

is expected. Figure 2 shows the evaluations of our data using (3) and suggesting $\nu=0.32\pm0.03$ and $z=1.34\pm0.04$. The accuracy of our data does not allow us to make a scaling plot for the finite-size shift in the position of the transition. In fact, it

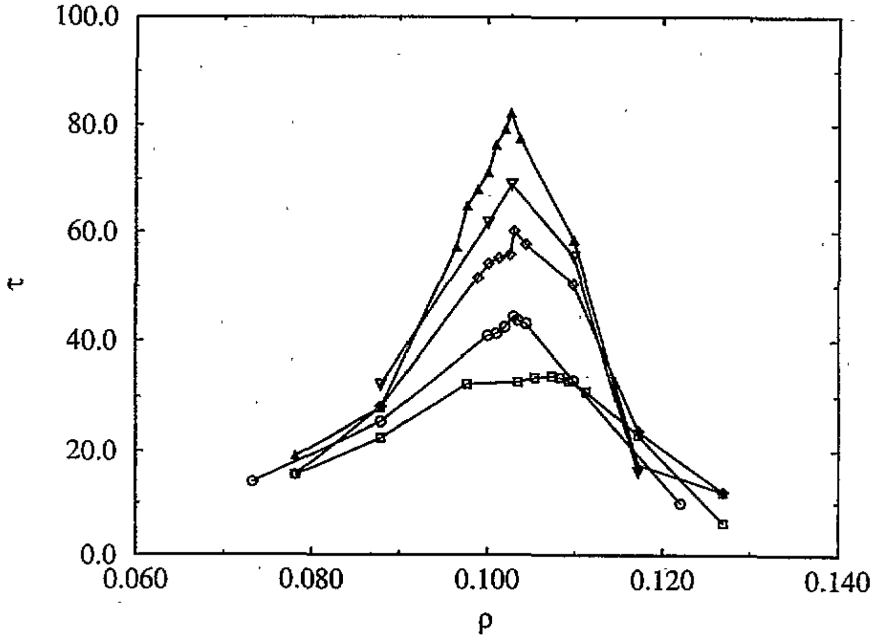

Figure 1. The relaxation time τ as a function of the density and for different system sizes: $L = 1024$ (□), 2048 (O), 4096 (◊), 5824 (▽) and 8192 (▲).

seems that this shift is rather small and we can conclude that the jamming point is at $\rho = 0.1028 \pm 0.0002$. Our results should be compared to the relaxation time analysis of a similar but deterministic model where an exponent $z = 1$ was obtained [7].

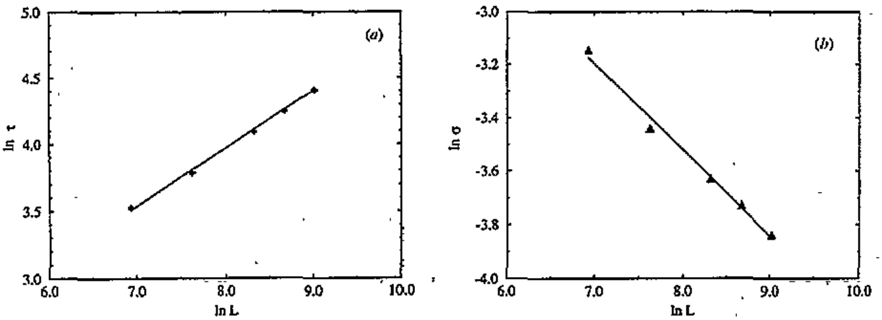

Figure 2. Finite size scaling of (a) the maximum of the relaxation time τ_m according to (3a) and (b) of the width σ of the curves in figure 1 at half heights evaluated using (3b).

We have localized the dynamical critical point by the divergence of the relaxation time and the obtained critical density is indeed near (though somewhat below) the position of the maximum in the fundamental diagram ($\rho_{\max} \approx 0.11$). If we want to compare this result with the percolation picture, which is a geometrical concept, we need to define the jammed regions. We chose a site-oriented definition: a site is considered to be in a jam, if two or more cars are within a window of five centred on the site. Other definitions can and have been used elsewhere [6, 8]. In particular, [6] defines a *car* to be jammed if it does not go with maximum speed. Going along with this definition, one might consider the percolation of 'sites' in the jammed state in 'car-space' rather than real space. However, such an approach does not alter our arguments qualitatively.

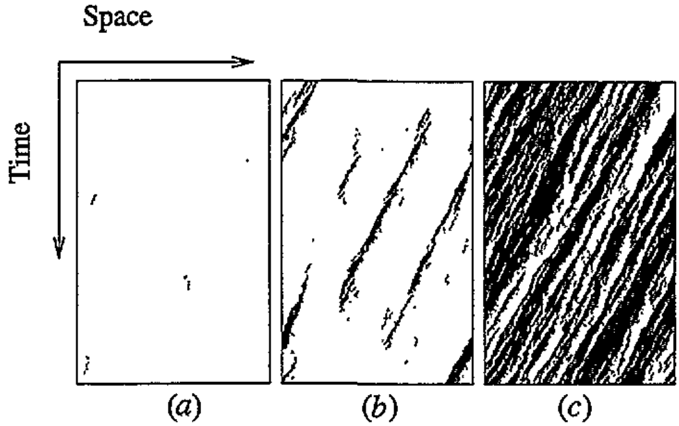

Figure 3. Snapshots of the jammed regions at three different densities: (*a*) $\rho = 0.1$, (*b*) $\rho = 0.15$ and (*c*) $\rho = 0.42$. Clearly, there is no percolation of waves in (*b*) although the density is already above the jamming transition.

Figure 3 shows the plots of jammed regions at three different densities: below the jamming transition (figure 3(*a*)), slightly above the transition (figure 3(*b*)) and at much higher density (figure 3(*c*)). We observe that the waves do not percolate immediately above the jamming transition but only at a second critical density which we denote by ρ_p.

In this way, we can speak about a geometrical transition in the system, when the waves become connected in spacetime, and an infinite wave appears. For this, we consider a (1 + 1)-dimensional (one space and one time dimension) directed percolation problem of waves starting at $t = 0$. By monitoring the number of samples n of size $L = 1000$ and $L = 10\,000$ where the waves survive until $t = 100\,000$ we estimate a percolation transition point $\rho_p \approx 0.36$ where n increases sharply. However, simple scaling assumptions do not seem to work for this correlated percolation problem. We have studied the average density of cars belonging to the considered waves as a function of time and no simple critical behaviour was observed. We also tried to locate ρ_p by calculating the average number $N(t)$ of waves starting at $t = 0$ and surviving at least until t. We averaged over at least 30 samples and used $L = 10\,000$ for every specified densities. At the percolation threshold $N(t)$ is expected to decay algebraically $N(t) \propto t^{-x}$. Based on figure 4 one would obtain $\rho_p \approx 0.41$ with $x \approx 0.3$, a value different from that obtained for a deterministic version of the model [6] ($x \approx 0.5$). Figure 4 does not look like a usual density plot for directed percolation where the critical curve separates lines bending downwards ($\rho < \rho_c$) from the lines saturating above the critical curve ($\rho > \rho_c$). The fact that the occurrence of the infinite waves does not coincide with the point where at least approximate scaling can be observed also shows that the percolation process has a complicated mechanism in this model. In any case, the percolation threshold is much above the dynamical jamming point.

The position of the percolation transition depends somewhat on the particular definition of the waves. The dynamic jamming point where τ diverges was defined without reference to the jams, but with any particular definition of waves, the actual ρ_p-value, where connectivity appears, depends strongly on p. If p is low, like in our case ($p = 0.25$), connectivity of jammed regions occurs only at much higher densities. Above this percolation critical point ρ_p, a wave of infinite size exists in the thermodynamical limit, i.e. a jam which is always

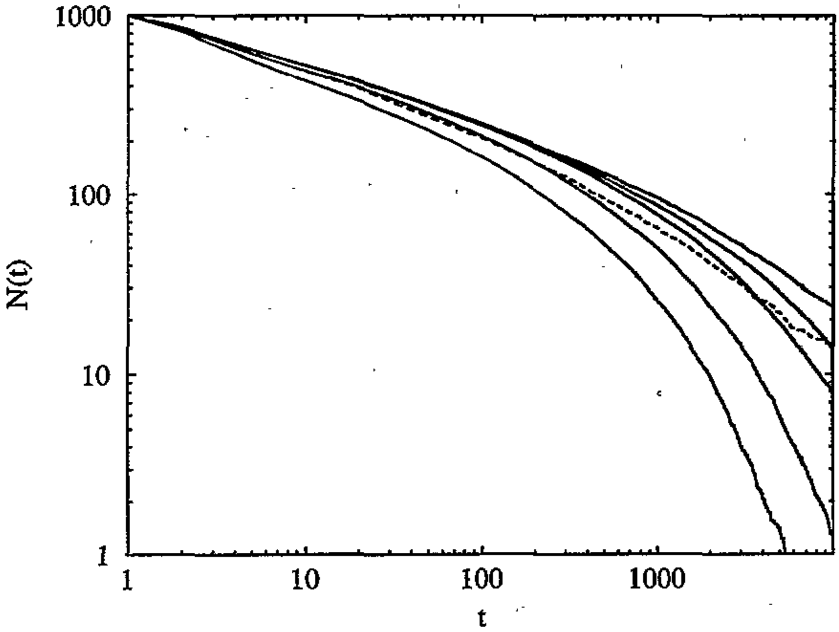

Figure 4. The average number $N(t)$ of waves surviving at least until t in a log–log plot for different densities $\rho = 0.25, 0.30, 0.35, 0.37, 0.41$ (full curves from bottom to top) and 0.43 (broken). The curves are normalized, so that at $t = 1$, $N(t) = 1000$. At $\rho \approx 0.41$ a scaling assumption is tolerable and a power law decay is obtained with an approximate exponent $x = 0.3$.

present. If p is large, connectivity occurs very close to the dynamical transition point. For $p = 0.5$ the percolation threshold ρ_p is barely distinguishable from ρ_c and this has lead to the assumption that the jamming transition can be considered as a short-range percolation problem [6]. Our result demonstrates that this is not necessarily so: for $p = 0.25$ the two transitions are clearly different.

The choice of $p = 0.25$ in our calculations was deliberate. The crucial point is that a much smaller value of p had to be chosen than $1/2$ where the dynamical jamming point and the percolation threshold become numerically very close. Thus we demonstrated that for small values of p the dynamical transition is not related to the short-range percolation properties. What is then the mechanism of cooperativity? Much below the threshold isolated jams occur and disappear and the cars leaving them move again in free flow until a fluctuation creates a new jam where they take a while. Above the jamming point (but below the percolation threshold) the density of jams is so high that the cars leaving one jam already face another one—possibly far away—which is fed by them. There is a considerable flux of cars from one jam to another, without the jams touching. This long-range transport between jams results in a non-local network for $\rho > \rho_c$ and just at ρ_c it takes a very (infinitely) long time to build it up. The flux of cars from one jam to the other one is also present above the percolation point. This results in a coarsening of waves and we think that this is the mechanism leading to the complicated percolation mechanism.

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